THE LEVEL DENSITIES OF RANDOM MATRIX UNITARY ENSEMBLES AND THEIR PERTURBATION INVARIABILITY

WANG ZHENGDONG[1] AND YAN KUIHUA[1,2]

Abstract. Using operator methods, we generally present the level densities for kinds of random matrix unitary ensembles in weak sense. As a corollary, the limit spectral distributions of random matrices from Gaussian, Laguerre and Jacobi unitary ensembles are recovered. At the same time, we study the perturbation invariability of the level densities of random matrix unitary ensembles. After the weight function associated with the 1-level correlation function is appended a polynomial multiplicative factor, the level density is invariant in the weak sense.

1. Introduction

In classical quantum mechanics, the statistical properties of energy levels can be described by the k-level correlation functions defined as (see Mehta[13])

$$R^k_{n\beta}(x_1, x_2, \cdots, x_k) = \frac{n!}{(n-k)!}\int\cdots\int P_{n\beta}(x_1, x_2, \cdots, x_n)\, dx_{k+1}\cdots dx_n,$$

where $P_{n\beta}(x_1, x_2, \cdots, x_n) = c_{n\beta}\cdot \exp(-\beta H)$ is the joint probability density function of n eigenvalues $x_1, x_2, \cdots, x_n$ of a $n\times n$ random matrix, $c_{n\beta}$ is the normalized constant. H is the Hamiltonian of the logarithmical interacting n particles system on a straight line, which is given by constraining one-body potential and logarithmic repulsive two-body potential, i.e.

$$H = \sum_{i=1}^{n} V(x_i) - \sum_{1\le i<j\le n} \log|x_i - x_j|, \quad x_i \in \mathbb{R}.$$

In general, $V(x)$ is called potential function and β Dyson's index. $\beta = 1, 2$ and 4 are corresponding to the orthogonal, unitary and symplectic ensembles respectively. When $k = 1$, $R^1_{n\beta}(x)$ can be explained as the distribution density of energy levels which can be found near by x. The level density denoted by $\sigma_\beta(x)$, which is a global quantity, is defined by the limit of the 1-level correlation function $R^1_{n\beta}(x)$. Then how to determine the level density? It can be traced back to Wigner's pioneering work[20, 21]. The results of early work are reviewed in [13, 16]. Recently, there are many

authors to concentrate on this problem (See Spohn[17], Bai and Yin[2], Nagao and Wadati[14], Haagerup and Thorbjørnsen[9], Girko[8], Kiessling and Spohn[10], Dueñez[4], Ledoux[11], etc.).

Notice that given different or special one-body potential $V(x)$, it will exhibited kinds of images for us. As a matter of fact, in the case of classical Gaussian ensembles, $V(x) = \dfrac{x^2}{2}$, $x \in \mathbb{R}$. The level density is the famous "semicircle law" first derived by Wigner[20, 21], i.e.

$$\sigma_{\beta}(x) = \begin{cases} \frac{1}{\pi}\sqrt{2n - x^2} & x^2 \leq 2n \\ 0 & x^2 \geq 2n. \end{cases}$$

In addition, in the case of Laguerre ensembles, the level density can be evaluated by a physical argument (See Bronk[3]), i.e.

$$\sigma_{\beta}(x) = \begin{cases} \frac{1}{\pi\sqrt{x}}\sqrt{2n - x} & 0 < x \leq 2n \\ 0 & x \geq 2n. \end{cases}$$

In the case of Jacobi ensembles, the level density can also be evaluated by a physical argument (See Leff[12]), i.e.

$$\sigma_{\beta}(x) = \begin{cases} \frac{n}{\pi\sqrt{1-x^2}} & -1 < x < 1 \\ 0 & otherwise. \end{cases}$$

In the case of unitary ensembles, $R^1_{n2}(x)$ can be expressed to a concise formula (see [13], [14]) which is closely correlated to classical orthogonal polynomials, i.e.

$$R^1_{n2}(x) = \sum_{m=0}^{n-1} p^2_m(x)\varpi(x), \tag{1}$$

where $\varpi(x) = \bar{c}\exp(-2V(x))$, $\bar{c}$ is the normalized constant and $p_m(x)$ be the m-order normalized orthogonal polynomials associated with the normalized weight function $\varpi(x)$, i.e. $\displaystyle\int p_m(x)p_n(x)\cdot\varpi(x)\,dx = \delta_{mn}$.

In [9], Haagerup and Thorbjørnsen only studied the Gaussian unitary ensemble(GUE) which was denoted by $SGRM(n, \sigma^2)$ there. Using the following property of Hermite polynomials

$$H_k(x + a) = \sum_{j=0}^{k} C^j_k(2a)^{k-j}H_j(x), \quad a \in \mathbb{R},$$

the authors directly obtained an equality for the complex Laplace transform of $\frac{1}{n}R^1_{n2}(x)$, i.e.

$$\int_{\mathbb{R}} \exp(sx)\Big(\frac{1}{n}R^1_{n2}(x)\Big)\,dx = \int_{\mathbb{R}} \exp(sx)\Big(\frac{1}{n\sqrt{\pi}}\sum_{k=0}^{n-1}\hat{H}^2_k(x)e^{-x^2}\Big)\,dx$$
$$= \exp(\frac{s^2}{4})\Phi(1-n,\,2;\,-\frac{s^2}{2}),\quad s\in\mathbb{C},$$

where $\hat{H}_k(x)$ is the k-order normalized Hermite polynomial, $\Phi(a,\,c;\,x)$ is the confluent hyper-geometric function(CHGF) with parameters a and c. Then they gave a short proof of Wigner's semicircle law.

As we know, the CHGFs are complicated series expansions. In [11], Ledoux pushed forward the investigation by Haagerup and Thorbjørnsen and only concentrated on the differential aspects of CHGFs. The author constructed an abstract framework of Markov diffusion generators, and in which derived the basic differential equations on Laplace transforms of $p^2_m(x)\varpi(x)$.

Using the recurrence formula of Hermite polynomials and the uniform integrability of random variable sequence, by the obtained differential equation, the author showed that

$$\lim_{m\to\infty}\int f\Big(\frac{x}{2\sqrt{m}}\Big)\frac{1}{m}\sum_{k=0}^{m-1}p^2_k(x)\varpi(x)\,dx = E\Big(f(\sqrt{X}\,Y)\Big),\text{for all } f\in C_b(\mathbb{R})$$

which determines the level density of GUE in the weak sense, where X and Y are two independent random variables with the uniform distribution on $[0,\,1]$ and the arcsine distribution on $(-1,\,+1)$ respectively. By the analogous technique, the author also obtained the level densities of Laguerre and Jacobi unitary ensembles respectively.

In this paper, we will generally deal with the problem for kinds of unitary ensembles (i.e. $\beta=2$) by operator method. It is no confusion to omit the subscript 2 in the below text. Moreover, as we will see, this method can effectively be used to study the perturbation invariability of level density.

It is well known that the normalized orthogonal polynomials $p_n(x)$ satisfy the following recursion formula (see section 2 for details)

$$xp_n(x)=\alpha_n p_{n+1}(x)+\beta_n p_n(x)+\gamma_n p_{n-1}(x),\quad n=1,\,2,\,3,\,\cdots.$$

In the paper, we assume that α_n and β_n satisfy the following **exponential growth conditions:**

$$\text{(2)}\qquad \alpha_n=\xi n^t(1+\xi_n),\quad \beta_n=\zeta n^t(1+\zeta_n)+\ \eta_n,$$

where $\xi\neq 0$, $\zeta\geq 0$, $0\leq t\leq 1$ are constants and $\lim_{n\to\infty}\xi_n=\lim_{n\to\infty}\zeta_n=\lim_{n\to\infty}\eta_n=\ 0.$

Note that the classical Hermite, Laguerre and Jacobi polynomials all satisfy exponential growth conditions (see section 2).

We define the "ascending","equilibrating" and "descending" operators as follow,

$$A_+p_n(x) = \alpha_n p_{n+1}(x), \quad A_0p_n(x) = \beta_n p_n(x), \quad A_-p_n(x) = \gamma_n p_{n-1}(x).$$

In section 3, we use these operators to obtain the moment inequality of the probability density $\sigma_n(x) = \frac{\sqrt{D_n}}{n}R_n^1(x\sqrt{D_n})$ and to show that the limit of k-th moment $M_n^{(k)}$ of $\sigma_n(x)$ exists (see theorem 3.2), where D_n is the 2nd moment of $\frac{1}{n}R_n^1(x)$. Further more, by considering the limit of characteristic functions of $\sigma_n(x)$, we obtain the main result in section 3, i.e. if there exists a probability density $\sigma(x)$ with k-th moments $M^{(k)} = \lim_{n\to\infty} M_n^{(k)}$, then (see theorem 3.3)

$$\sigma_n(x) \xrightarrow{w} \sigma(x), \quad n \to \infty,$$

where w means weak convergence.

In section 4, the perturbation invariability of level density is studied. We will consider another weight function $\hat{\varpi}(x)$ by appending a polynomial multiplicative factor $p^2(x)$ to $\varpi(x)$, which uniquely determines a family of orthonormal polynomials $\{\hat{p}_m(x)\}$. Accordingly, 1-level correlation function is $\hat{R}_n^1(x) = \sum_{j=0}^{n-1}\hat{p}_j^2(x)p^2(x)\varpi(x)$. We will show that the probability density $\hat{\sigma}_n(x) = \frac{\sqrt{D_n}}{n}\hat{R}_n^1(x\sqrt{D_n})$ is still weakly convergent to $\sigma(x)$.

2. Orthogonal Polynomials

In this section, we briefly introduce the orthogonal polynomials. We would like to lead the readers to refer [5, 15, 18]. Let us start by quoting a classical result as follow.

Theorem 2.1. *The following relation holds for the normalized orthogonal polynomials $p_n(x)$ with normalized weight function $\varpi(x)$:*

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \quad n = 1, 2, 3, \cdots$$

where α_n, β_n and γ_n are constants which can be expressed in terms of the coefficients a_n and b_n of the highest terms in $p_n(x) = a_nx^n + b_nx^{n-1}\cdots$, $a_n \neq 0$, i.e.

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad n = 0, 1, \cdots,$$

$$\gamma_n = \alpha_{n-1}, \quad \gamma_0 = 0, \quad n = 1, 2, \cdots.$$

If we regard the multiplication by x as an operator A_x, clearly the operator A_x plays "ascending","equilibrating" and "descending" roles when it acts on the n-order orthogonal polynomial $p_n(x)$, i.e.

$$A_x = A_+ + A_0 + A_-,$$

where A_+, A_0 and A_- are called "ascending","equilibrating" and "descending" operators respectively and defined by

$$A_+ p_n(x) = \alpha_n p_{n+1}(x), \quad n = 0,\, 1,\, \cdots,$$

$$A_0 p_n(x) = \beta_n p_n(x), \quad n = 0,\, 1,\, \cdots, \tag{3}$$

$$A_- p_n(x) = \gamma_n p_{n-1}(x), \quad A_- p_0(x) = 0, \quad n = 1,\, 2,\, \cdots.$$

In the below text, we always assume that α_n and β_n satisfies the exponential growth conditions (2). Let us consider several classical examples.

1) Hermite polynomials

The weight function

$$w(x) = \exp(-x^2), \quad x \in \mathbb{R}$$

and

$$\tilde{H}_n(x) \equiv \frac{1}{\sqrt{2^n n!}} H_n(x),$$

where

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n}(\exp(-x^2))$$

is the standard n-th Hermite polynomial which satisfies the orthogonal relation

$$\pi^{-\frac{1}{2}} \int_{\mathbb{R}} H_k(x) H_j(x) \cdot w(x)\, dx = 2^k k! \delta_{kj}.$$

$\tilde{H}_n(x)$ satisfies the following recurrence relation

$$x\tilde{H}_n(x) = \sqrt{\frac{n+1}{2}}\tilde{H}_{n+1}(x) + \sqrt{\frac{n}{2}}\tilde{H}_{n-1}(x), \quad n = 1,\, 2,\, 3,\, \cdots$$

$$\tilde{H}_0(x) = 1, \quad \tilde{H}_1(x) = \sqrt{2}\, x.$$

Thus $\alpha_n = \sqrt{\frac{n}{2}}$, $\beta_n = 0$, i.e. $\xi = \frac{1}{\sqrt{2}}$, $\zeta = 0$, $t = \frac{1}{2}$, $\xi_n = \zeta_n = \eta_n = 0$.

2) Laguerre polynomials

The weight function

$$w^{(a)}(x) = x^a e^{-x}, \quad x \in (0,\, \infty)$$

and

$$\tilde{L}_n^{(a)}(x) \equiv \sqrt{\frac{\Gamma(k+1)}{\Gamma(k+a+1)}} L_n^{(a)}(x),$$

where

$$L_n^{(a)}(x) = \frac{x^{-a}e^x}{n!}\frac{d^n}{dx^n}(x^{n+a}e^{-x})$$

is the standard n-th Laguerre polynomial which satisfies the orthogonal relation

$$\int_0^\infty L_k^{(a)}(x)L_j^{(a)}(x)\cdot x^a e^{-x}\,dx = \frac{\Gamma(k+a+1)}{\Gamma(k+1)}\delta_{kj}.$$

$\tilde{L}_n^{(a)}(x)$ satisfies the following recurrence relation

$$x\tilde{L}_n^{(a)}(x) = -\sqrt{(n+1)(n+1+a)}\tilde{L}_{n+1}^{(a)}(x)+(2n+a+1)\tilde{L}_n^{(a)}(x)-\sqrt{n(n+a)}\tilde{L}_{n-1}^{(a)}(x),$$

$$\tilde{L}_0^{(a)}(x) = \frac{1}{\sqrt{\Gamma(a+1)}}, \quad \tilde{L}_1^{(a)}(x) = \frac{-x+a+1}{\sqrt{\Gamma(a+2)}}, \quad n = 1,\,2,\,3,\,\cdots.$$

Thus $\alpha_n = -\sqrt{(n+1)(n+1+a)}$, $\beta_n = 2n+a+1$, i.e. $\xi = -1$, $\zeta = 2$, $t = 1$, $\xi_n = \big(1+\frac{2+a}{n}+\frac{1+a}{n^2}\big)^{1/2}-1$, $\zeta_n = \frac{1+a}{2n}$, $\eta_n = 0$.

3) Jacobi polynomials

The weight function

$$w^{(a,\,b)}(x) = (1-x)^a(1+x)^b, \quad x\in(-1,\,1)$$

and

$$\tilde{J}_n^{(a,\,b)}(x) \equiv (c_n^{(a,\,b)})^{-\frac{1}{2}}J_n^{(a,\,b)}(x),$$

where

$$c_k^{(a,\,b)} = \frac{2^{a+b+1}}{2k+a+b+1}\frac{\Gamma(k+a+1)\Gamma(k+b+1)}{\Gamma(k+1)\Gamma(k+a+b+1)}$$

and

$$J_n^{(a,\,b)}(x) = \frac{1}{(1-x)^a(1+x)^b}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\big((1-x)^{n+a}(1+x)^{n+b}\big)$$

is the standard n-th Jacobi polynomial which satisfies the orthogonal relation

$$\int_{-1}^1 J_k^{(a,\,b)}(x)J_j^{(a,\,b)}(x)\cdot(1-x)^a(1+x)^b\,dx = c_k^{(a,\,b)}\delta_{kj}.$$

$\tilde{J}_n^{(a,\,b)}(x)$ satisfies the following recurrence relation

$$\begin{aligned} x\tilde{J}_n^{(a,\,b)}(x) =& \frac{2}{2n+a+b+2}\sqrt{\frac{(n+1)(n+a+1)(n+b+1)(n+a+b+1)}{(2n+a+b+3)(2n+a+b+1)}}\tilde{J}_{n+1}^{(a,\,b)}(x) \\ &- \frac{a^2-b^2}{(2n+a+b)(2n+a+b+2)}\tilde{J}_n^{(a,\,b)}(x) \\ &+ \frac{2}{2n+a+b}\sqrt{\frac{n(n+a)(n+b)(n+a+b)}{(2n+a+b+1)(2n+a+b-1)}}\tilde{J}_{n-1}^{(a,\,b)}(x) \end{aligned}$$

Thus $\xi = \frac{1}{2}$, $\zeta = 0$, $t = 0$. ξ_n, ζ_n and η_n have their respective expressions.

In particular, if $a = b = 0$, then it is the Legendre polynomial. The recurrence relation is

$$x\tilde{J}_n^{(0,\,0)}(x) = \frac{n+1}{\sqrt{(2n+3)(2n+1)}}\tilde{J}_{n+1}^{(0,\,0)}(x) + \frac{n}{\sqrt{(2n+1)(2n-1)}}\tilde{J}_{n-1}^{(0,\,0)}(x).$$

3. The Level Densities

3.1. **Main Results.** In this section, we consider the limit of the global 1-level correlation functions $R_n^1(x) = \sum_{m=0}^{n-1} p_m^2(x)\varpi(x)$ for unitary ensembles, where $p_m(x)$ is the m-th normalized orthogonal polynomial associated with the normalized weight function $\varpi(x) = \bar{c}\exp(-2V(x))$, $\bar{c}$ is the normalized constant.

Specially, for the Gaussian unitary ensemble (GUE), $V(x) = \frac{x^2}{2}$, $x \in \mathbb{R}$,

$$\varpi(x) = \pi^{-\frac{1}{2}}w(x), \quad p_m(x) = \tilde{H}_m(x), \quad x \in \mathbb{R},$$

where $\tilde{H}_m(x)$ is the m-th normalized orthogonal Hermite polynomial associated with the weight function $w(x) = \exp(-x^2)$.

For Laguerre unitary ensemble (LAUE), $V(x) = \begin{cases} -a\log x + x & 0 < x < \infty \\ \infty & otherwise \end{cases}$,

$$\varpi(x) = \bar{c}w^{(2a)}(2x), \quad p_m(x) = \sqrt{2/\bar{c}}\tilde{L}_m^{(2a)}(2x), \quad x \in (0,\, \infty),$$

where $\tilde{L}_m^{(a)}(x)$ is the m-th normalized orthogonal Laguerre polynomial associated with the weight function $w^{(a)}(x) = x^a e^{-x}$.

For Jacobi unitary ensemble (JUE), $V(x) = \begin{cases} -\log\big((1-x)^a(1+x)^b\big) & -1 < x < 1 \\ \infty & otherwise \end{cases}$,

$$\varpi(x) = \bar{c}w^{(2a,\,2b)}(x), \quad p_m(x) = \sqrt{1/\bar{c}}\tilde{J}_m^{(2a,\,2b)}(x), \quad x \in (-1,\, 1),$$

where $\tilde{J}_m^{(a,\,b)}(x)$ is the m-th normalized orthogonal Hermite polynomial associated with the weight function $w^{(a,\,b)}(x) = (1-x)^a(1+x)^b$. If $a = b = 0$, it is the Legendre unitary ensemble (LEUE).

Now it is no less of generality to consider the whole real axis $\mathbb{R}$. Denoted by D_n the 2nd moment of the probability density $\frac{1}{n}R_n^1(x)$. Then

$$\begin{aligned}D_n &= \frac{1}{n}\int_{\mathbb{R}} x^2 R_n^1(x)\,dx = \frac{1}{n}\sum_{j=0}^{n-1}\langle xp_j,\, xp_j\rangle \\ &= \frac{1}{n}\sum_{j=0}^{n-1}(\alpha_j^2+\beta_j^2+\gamma_j^2).\end{aligned}$$

But $\gamma_j = \alpha_{j-1}$, hence

$$D_n = \frac{1}{n}\sum_{j=0}^{n-2}(2\alpha_j^2+\beta_j^2) + \frac{1}{n}(\alpha_{n-1}^2+\beta_{n-1}^2). \tag{4}$$

Let

$$\sigma_n(x) = \frac{\sqrt{D_n}}{n}R_n^1(x\sqrt{D_n}).$$

Then the k-th moment $M_n^{(k)}$ of probability density $\sigma_n(x)$ is

$$\begin{aligned}M_n^{(k)} &= \int_{\mathbb{R}} x^k\sigma_n(x)\,dx = \frac{1}{n\cdot(D_n)^{k/2}}\sum_{j=0}^{n-1}\int_{\mathbb{R}} x^k p_j^2(x)\varpi(x)\,dx \\ &= \frac{1}{n\cdot(D_n)^{k/2}}\sum_{j=0}^{n-1}\Big\langle x^k p_j,\, p_j\Big\rangle_{L^2(\varpi)} \\ &= \frac{1}{n\cdot(D_n)^{k/2}}\sum_{j=0}^{n-1}\Big\langle (A_+ + A_0 + A_-)^k p_j,\, p_j\Big\rangle_{L^2(\varpi)}.\end{aligned}$$

Let Λ_k^i be an operator set composed of those terms in the expansion of $(A_+ + A_0 + A_-)^k$, in which the operators A_+ and A_- exactly appear i times. And note that for all $T \notin \bigcup_i \Lambda_k^i$, $\langle Tp_j,\, p_j\rangle_{L^2(\varpi)} = 0$, thus we obtain the following lemma

Lemma 3.1.

$$M_n^{(k)} = \frac{1}{n\cdot(D_n)^{k/2}}\sum_{j=0}^{n-1}\sum_{i=0}^{[\frac{k}{2}]}\sum_{T\in\Lambda_k^i}\Big\langle Tp_j,\, p_j\Big\rangle_{L^2(\varpi)}. \tag{5}$$

Moreover, set $r_1(k) = \sum_{i=0}^{[\frac{k}{2}]} C_k^i C_{k-i}^i \xi^{2i} \zeta^{k-2i}$, $r_2(k) = (2\xi^2 + \zeta^2)^{k/2}$ and

$$M^{(k)} = \frac{r_1(k)}{r_2(k)} \cdot \frac{(2t+1)^{k/2}}{(kt+1)},$$

then we have the following limit theorem for the k-th moment $M_n^{(k)}$.

Theorem 3.2. *Under the exponential growth conditions,*

$$\lim_{n\to\infty} M_n^{(k)} = M^{(k)}, \quad \textit{for any } k \in \mathbb{Z}^+. \tag{6}$$

The proof is based on elaborate estimations of $M_n^{(k)}$. Here we firstly consider the classical cases instead of being anxious to verify this theorem.

Example. For GUE, $\xi = \frac{1}{\sqrt{2}}$, $\zeta = 0$, $t = \frac{1}{2}$, so $M^{(2m)} = \frac{C_{2m}^m}{m+1}$, $M^{(2m+1)} = 0$. Moreover, $M^{(k)}$ is exactly the k-th moment of density function

$$\sigma_G(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & x^2 \le 4 \\ 0 & x^2 \ge 4. \end{cases}$$

For LAUE, $\xi = -1$, $\zeta = 2$, $t = 1$, so $M^{(k)} = \frac{C_{2k}^k}{2^{k/2}(k+1)}$. Moreover, $M^{(k)}$ is exactly the k-th moment of density function

$$\sigma_L(x) = \begin{cases} \frac{\sqrt{2\sqrt{2}-x}}{\pi\sqrt{2x}} & 0 < x \le 2\sqrt{2} \\ 0 & x > 2\sqrt{2}. \end{cases}$$

For JUE, $\xi = \frac{1}{2}$, $\zeta = 0$, $t = 0$, so $M^{(2m)} = \frac{C_{2m}^m}{2^m}$, $M^{(2m+1)} = 0$. Moreover, $M^{(k)}$ is accordingly the k-th moment of density function

$$\sigma_J(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad -1 < x < 1.$$

Now suppose that there exists a probability density $\sigma(x)$ with k-th moments $M^{(k)}$. Then we have

Theorem 3.3. *Under the exponential growth conditions,*

$$\sigma_n(x) \xrightarrow{w} \sigma(x), \quad n \to \infty. \tag{7}$$

where w means weak convergence. In general, $\sigma(x)$ is called a level density.

Corollary 3.4. *The level densities of Gauss, Laguerre and Jacobi unitary ensembles are $\sigma_G(x)$, $\sigma_L(x)$ and $\sigma_J(x)$ respectively in weak sense.*

By the above example and theorem 3.3, it is obvious.

3.2. **Proof of main results.** Now we begin to verify the main results. First of all, let us consider the 2nd moment D_n of $\frac{1}{n}R_n^1(x)$. If $\zeta > 0$, set

$$u_j = \max_{j-k\le m\le j+k}\Big\{|\xi_m|,\ |\zeta_m| + \frac{|\eta_m|}{\zeta m^t}\Big\}, \tag{8}$$

then $\lim_{j\to\infty} u_j = 0$. It is no less of generality to assume $u_j < 1$. Thus by the exponential growth conditions (2) and equality (4),

$$\frac{2\xi^2+\zeta^2}{n}\sum_{j=0}^{n-2} j^{2t}(1-u_j)^2 \le D_n \le \frac{2\xi^2+\zeta^2}{n}\sum_{j=0}^{n-1} j^{2t}(1+u_j)^2. \tag{9}$$

If $\zeta = 0$, one can easily choose another proper infinitesimal sequence u_j such that the above inequality is still valid. By (9), it is easy to see that

$$\frac{D_n}{n^{2t}} = O(1). \tag{10}$$

Proof of theorem 3.2. Firstly, we come to verify a result about series limit. That is for any sequence x_n and y_n, if $\lim_{n\to\infty} x_n = 0$ and $y_n > 0$, $\sum_{n=0}^{\infty} y_n = \infty$, then

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n} y_j x_j}{\sum_{j=0}^{n} y_j} = 0. \tag{11}$$

Indeed, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, as $n \ge N$, $|x_n| < \frac{\varepsilon}{2}$. Thus

$$\Big|\frac{\sum_{j=0}^{n-1} y_j x_j}{\sum_{j=0}^{n} y_j}\Big| \le \frac{\sum_{j=0}^{N} y_j |x_j|}{\sum_{j=0}^{n} y_j} + \frac{\varepsilon}{2}\cdot\frac{\sum_{j=N}^{n} y_j}{\sum_{j=0}^{n} y_j} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now let us set two cases to verify the theorem.

Case 1. $\zeta > 0$.

By exponential growth conditions (2) and the definition (8) of u_j, we see that

$$\max_{j-k\le m\le j+k}\{|\alpha_m|\} \le |\xi|(j+k)^t(1+u_j), \qquad \max_{j-k\le m\le j+k}\{\beta_m\} \le \zeta(j+k)^t(1+u_j). \tag{12}$$

Thus for $j > k$ and $T \in \Lambda^i_k$,

$$\langle Tp_j, p_j\rangle \le \Big(\max_{j-k\le m\le j+k}\{|\alpha_m|\}\Big)^{2i}\Big(\max_{j-k\le m\le j+k}\{\beta_m\}\Big)^{k-2i}$$
$$\le \Big(|\xi|(j+k)^t(1+u_j)\Big)^{2i}\Big(\zeta(j+k)^t(1+u_j)\Big)^{k-2i}.$$

And then

$$\langle Tp_j, p_j\rangle \le \xi^{2i}\zeta^{k-2i}(j+k)^{kt}(1+u_j)^k. \tag{13}$$

Therefore by lemma 3.1 and inequalities (13),(9)-(10),

$$M_n^{(k)} \le \frac{r_1(k)}{r_2(k)}\cdot\frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}(1+u_j)^k}{\Big(\frac{1}{n}\sum_{j=0}^{n-2}j^{2t}(1-u_j)^2\Big)^{k/2}}+O(\frac{1}{n^{kt+1}}),\quad \text{for } k\in\mathbb{Z}^+. \tag{14}$$

On the other hand, also by exponential growth conditions (2) and the definition (8) of u_j,

$$\min_{j-k\le m\le j+k}\{|\alpha_m|\}\ge|\xi|(j-k)^t(1-u_j),\qquad \min_{j-k\le m\le j+k}\{\beta_m\}\ge\zeta(j-k)^t(1-u_j). \tag{15}$$

Then for $j > k$ and $T \in \Lambda^i_k$,

$$\langle Tp_j, p_j\rangle \ge \Big(\min_{j-k\le m\le j+k}\{|\alpha_m|\}\Big)^{2i}\Big(\min_{j-k\le m\le j+k}\{\beta_m\}\Big)^{k-2i}$$
$$\ge \Big(\xi(j-k)^t(1-u_j)\Big)^{2i}\Big(\zeta(j-k)^t(1-u_j)\Big)^{k-2i}.$$

Therefore,

$$\langle Tp_j, p_j\rangle \ge \xi^{2i}\zeta^{k-2i}(j-k)^{kt}(1-u_j)^k. \tag{16}$$

Thus by lemma 3.1 and inequalities (9)-(10) and (16),

$$M_n^{(k)} \ge \frac{r_1(k)}{r_2(k)}\cdot\frac{\frac{1}{n}\sum_{j=k}^{n-1}(j-k)^{kt}(1-u_j)^k}{\Big(\frac{1}{n}\sum_{j=0}^{n-1}j^{2t}(1+u_j)^2\Big)^{k/2}}+O(\frac{1}{n^{kt+1}}),\quad \text{for } n>k. \tag{17}$$

Now let us come to consider the limit of the dexter series in inequalities (14) and (17). By equality (11),

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}(1+u_j)^k}{\Big(\frac{1}{n}\sum_{j=0}^{n-2}j^{2t}(1-u_j)^2\Big)^{k/2}}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}}{\Big(\frac{1}{n}\sum_{j=0}^{n-2}j^{2t}\Big)^{k/2}}=\frac{(2t+1)^{k/2}}{(kt+1)} \tag{18}$$

and

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=K_0}^{n-1}(j-k)^{kt}(1-u_j)^k}{\Big(\frac{1}{n}\sum_{j=0}^{n-1}j^{2t}(1+u_j)^2\Big)^{k/2}}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{j=0}^{n-1}(j-k)^{kt}}{\Big(\frac{1}{n}\sum_{j=0}^{n-2}j^{2t}\Big)^{k/2}}=\frac{(2t+1)^{k/2}}{(kt+1)}. \tag{19}$$

Then by inequalities (14) and (17) and equalities (18)-(19), we have

$$\lim_{n\to\infty}M_n^{(k)}=\frac{r_1(k)}{r_2(k)}\cdot\frac{(2t+1)^{k/2}}{(kt+1)}=M^{(k)},\quad \text{if } \zeta>0. \tag{20}$$

Case 2. $\zeta=0$.

Set $v_j=\max_{j-k\le m\le j+k}\big\{|\xi_m|,\,|\eta_m|\big\}$. It is obvious that $\lim_{j\to\infty}v_j=0$. So it is no less of generality to assume that $v_j<1$. Thus by exponential growth conditions (2), for $0\le j-k\le m\le j+k$,

$$|\xi|(j-k)^t(1-v_j)\le|\alpha_m|\le|\xi|(j+k)^t(1+v_j),\quad |\beta_m|\le v_j. \tag{21}$$

Of course, v_j may be identity to zero. So we come to discuss the estimation of $M_n^{(k)}$ in terms of the parity of k.

(i) If k is odd, then for $j>k$ and $T\in\Lambda_k^i$,

$$\begin{aligned}\Big|\langle Tp_j,\,p_j\rangle\Big|&\le\Big(\xi(j+k)^t(1+v_j)\Big)^{2i}v_j^{k-2i}\\&\le\xi^{2i}v_j^{k-2i}(j+k)^{kt}(1+v_j)^k.\end{aligned}$$

Let $\bar v_j=\sum_{i=0}^{[\frac{k}{2}]}C_k^iC_{k-i}^i\xi^{2i}v_j^{k-2i}$, thus by inequality (9)-(10) and lemma 3.1,

$$\Big|M_n^{(k)}\Big|\le\frac{1}{r_2(k)}\cdot\frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}(1+v_j)^k\bar v_j}{\Big(\frac{1}{n}\sum_{j=0}^{n-2}j^{2t}(1-u_j)^2\Big)^{k/2}}+O(\frac{1}{n^{kt+1}}),\quad \text{for } k\in\mathbb{Z}^+. \tag{22}$$

Note that $\lim_{j\to\infty} \bar{v}_j = \lim_{j\to\infty} v_j = 0$. Then by equality (11) and (18),

$$\lim_{n\to\infty} \frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}(1+v_j)^k \bar{v}_j}{\Big(\frac{1}{n}\sum_{j=0}^{n-2} j^{2t}(1-u_j)^2\Big)^{k/2}} = 0.$$

Thus by inequalities (22),

$$\lim_{n\to\infty} M_n^{(k)} = 0 = M^{(k)} \text{ (here } r_1(k) = 0). \tag{23}$$

(ii) If k is even, then by inequality (21), for $j > k$ and $T \in \Lambda_k^i,\ i \neq \frac{k}{2}$,

$$\Big|\langle Tp_j,\, p_j\rangle\Big| \leq \xi^{2i} v_j^{k-2i}(j+k)^{kt}(1+v_j)^k \tag{24}$$

and for $T \in \Lambda_k^{k/2}$,

$$\xi^k (j-k)^{kt}(1-v_j)^k \leq \langle Tp_j,\, p_j\rangle \leq \xi^k (j+k)^{kt}(1+v_j)^k. \tag{25}$$

Set

$$\tilde{M}_n^{(k)} = \frac{1}{n\cdot(D_n)^{k/2}} \sum_{j=0}^{n-1}\sum_{i\neq\frac{k}{2}}\sum_{T\in\Lambda_k^i} \Big\langle Tp_j,\, p_j\Big\rangle$$

and

$$\bar{M}_n^{(k)} = \frac{1}{n\cdot(D_n)^{k/2}} \sum_{j=0}^{n-1}\sum_{T\in\Lambda_k^{k/2}} \Big\langle Tp_j,\, p_j\Big\rangle,$$

then by equality (5),

$$M_n^{(k)} = \tilde{M}_n^{(k)} + \bar{M}_n^{(k)}.$$

It is completely analogous with the above discussion in the case of odd k, we see that by inequalities (9) and (24) and equality (11),

$$\lim_{n\to\infty} \tilde{M}_n^{(k)} = 0. \tag{26}$$

But with respect to $\bar{M}_n^{(k)}$, by inequalities (9) and (25), we have

$$\bar{M}_n^{(k)} \leq \frac{C_k^{k/2}\xi^k}{r_2(k)} \cdot \frac{\frac{1}{n}\sum_{j=0}^{n-1}(j+k)^{kt}(1+v_j)^k}{\Big(\frac{1}{n}\sum_{j=0}^{n-2} j^{2t}(1-u_j)^2\Big)^{k/2}} + O(\frac{1}{n^{kt+1}}) \tag{27}$$

and

$$\bar{M}_n^{(k)} \geq \frac{C_k^{k/2}\xi^k}{r_2(k)} \cdot \frac{\dfrac{1}{n}\displaystyle\sum_{j=k}^{n-1}(j-k)^{kt}(1-v_j)^k}{\Big(\dfrac{1}{n}\displaystyle\sum_{j=0}^{n-2} j^{2t}(1+u_j)^2\Big)^{k/2}} + O(\frac{1}{n^{kt+1}}). \tag{28}$$

Then by equalities (18)-(19) and inequalities (27)-(28),

$$\lim_{n\to\infty} \bar{M}_n^{(k)} = \frac{C_k^{k/2}\xi^k}{r_2(k)} \cdot \frac{(2t+1)^{k/2}}{(kt+1)}. \tag{29}$$

Therefore by equalities (26) and (29),

$$\lim_{n\to\infty} M_n^{(k)} = \frac{C_k^{k/2}\xi^k}{r_2(k)} \cdot \frac{(2t+1)^{k/2}}{(kt+1)} = \frac{r_1(k)}{r_2(k)} \cdot \frac{(2t+1)^{k/2}}{(kt+1)} = M^{(k)}. \tag{30}$$

Combining equalities (20), (23) and (30), we complete the proof of the theorem 3.2.

Next we come to verify the theorem 3.3. In the first place, we have the following rough estimation of k-th moment $M_n^{(k)}$.

Proposition 3.5. *Under the exponential growth conditions (2), for any $\varepsilon > 0$ there exist an integer $K_0(\varepsilon)$ independent on k and n such that*

$$\Big|M_n^{(k)}\Big| \leq d_0^k(\varepsilon k)^{kt}, \quad \textit{for } k, n \geq K_0(\varepsilon), \tag{31}$$

where d_0 is a constant independent on k, n and ε.

Proof. By the exponential growth conditions (2), there is a constant d_1 independent on k and n, such that

$$|\alpha_n| \leq d_1 n^t, \quad |\beta_n| \leq d_1 n^t.$$

Then for any $T \in \Lambda_k^i$,

$$\Big|\langle Tp_j,\, p_j\rangle\Big| \leq d_1^k(j+k)^{kt}. \tag{32}$$

And by inequality (9), there is a constant $d_2 > 0$ independent on k and n, such that

$$D_n \geq d_2 n^{2t}. \tag{33}$$

Thus by lemma 3.1 and inequalities (32)-(33),

$$\Big|M_n^{(k)}\Big| \leq \frac{\displaystyle\sum_{i=0}^{[\frac{k}{2}]} C_k^i C_{k-i}^i}{(d_2)^{k/2}} \cdot \frac{\displaystyle\sum_{j=0}^{n-1} d_1^k(j+k)^{kt}}{n^{kt+1}} \leq \frac{d_1^k \displaystyle\sum_{i=0}^{[\frac{k}{2}]} C_k^i C_{k-i}^i}{(d_2)^{k/2}} \cdot \Big(\frac{n+k}{n}\Big)^{kt}$$

But $\sum_{i=0}^{[\frac{k}{2}]} C_k^i C_{k-i}^i \leq 3^k$. Set $d_0 = \dfrac{3d_1}{d_2^{1/2}}$, then

$$\left| M_n^{(k)} \right| \leq d_0^k \Big(1 + \frac{k}{n} \Big)^{kt}.$$

Note that for all $\varepsilon > 0$, set $K_0(\varepsilon) = [\frac{2}{\varepsilon}] + 2$, then when $k, n \geq K_0(\varepsilon)$, $1 + \dfrac{k}{n} \leq \dfrac{\varepsilon}{2} k + \dfrac{\varepsilon}{2} k \leq \varepsilon k$. Consequently,

$$\left| M_n^{(k)} \right| \leq d_0^k (\varepsilon k)^{kt}, \quad \text{for } k, n \geq K_0(\varepsilon).$$

Thus we complete the proof of proposition 3.5.

Proof of theorem 3.3. Let $f_n(\theta)$ and $f(\theta)$ be the characteristic functions of $\sigma_n(x)$ and $\sigma(x)$ respectively. It is sufficient to show that

$$\lim_{n \to \infty} f_n(\theta) = f(\theta).$$

Note that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{|\theta|^k}{k!} \int |x|^k \sigma_n(x)\, dx &= \sum_{m=0}^{\infty} \frac{|\theta|^{2m}}{(2m)!} M_n^{(2m)} + \sum_{m=0}^{\infty} \frac{|\theta|^{2m+1}}{(2m+1)!} E(|X_n|^{2m+1}) \\ &\leq \sum_{m=0}^{\infty} \frac{|\theta|^{2m}}{(2m)!} M_n^{(2m)} + \sum_{m=0}^{\infty} \frac{|\theta|^{2m+1}}{(2m+1)!} \Big(1 + M_n^{(2m+2)} \Big). \end{aligned}$$

Then by the rough estimation (31) of $M_n^{(k)}$ in proposition 3.5, for any θ we can choose $\varepsilon > 0$ such that

$$\sum_{k=0}^{\infty} \frac{|\theta|^k}{k!} \int |x|^k \sigma_n(x)\, dx < \infty, \quad \text{for } n \geq K_0(\varepsilon).$$

Thus we have

$$\begin{aligned} f_n(\theta) &= \int e^{i\theta x} \sigma_n(x)\, dx = \int \sum_{k=0}^{\infty} \frac{(i\theta x)^k}{k!} \sigma_n(x)\, dx \\ &= \sum_{k=0}^{\infty} \int \frac{(i\theta x)^k}{k!} \sigma_n(x)\, dx = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} M_n^{(k)}, \quad \text{for } n \geq K_0(\varepsilon). \end{aligned}$$

Analogously, it can be verified that $f(\theta) = \sum_{k=0}^{\infty} \dfrac{(i\theta)^k}{k!} M^{(k)}$.

Note that $0 \leq t \leq 1$. So for any θ we can choose $\varepsilon > 0$ such that

$$\sum_{k=K_0(\varepsilon)}^{\infty} \frac{|\theta d_0|^k (\varepsilon k)^{kt}}{k!} < \infty.$$

Therefore by theorem 3.2, inequality (31) and Lebesgue control convergent theorem for series,

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}M_{n}^{(k)}=\sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}\lim_{n\to\infty}M_{n}^{(k)}=\sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}M^{(k)}.$$

Thus we obtain

$$\lim_{n\to\infty}f_n(\theta)=f(\theta).$$

It completes the proof of theorem 3.3.

4. Perturbation Invariability

Now Let $p(x)$ be a fixed l-order polynomial, denoted by $\hat{p}_j(x)$ the j-th normalized orthogonal polynomials associated with the weight function $\hat{\varpi}(x)=p^2(x)\varpi(x)$. By equality (1), the 1-level correlation function is

$$\hat{R}_{n}^{1}(x)=\sum_{j=0}^{n-1}\hat{p}_{j}^{2}(x)p^2(x)\varpi(x).$$

Let

$$\hat{\sigma}_n(x)=\frac{\sqrt{D_n}}{n}\hat{R}_{n}^{1}(x\sqrt{D_n}).$$

Set

$$\mathcal{H}_n=span\{p_0(x),\, p_1(x),\, \cdots,\, p_{n-1}(x)\},$$

then $\mathcal{H}_n$ is a subspace of $L^2(\mathbb{R},\,\varpi(x)dx)$ with n dimensions. It is obvious that $\{\hat{p}_0(x)p(x),\,\cdots,\,\hat{p}_{n-l-1}(x)p(x)\}$ is a family of normalized orthogonal vectors in $\mathcal{H}_n$. We can extend this set of vectors, such that it makes up of a normalized orthogonal base of $\mathcal{H}_n$. Denoted it by

$$\{e_{0}^{(n)}(x),\,\cdots,\,e_{l-1}^{(n)}(x),\,\hat{p}_0(x)p(x),\,\cdots,\,\hat{p}_{n-l-1}(x)p(x)\}.$$

Let P_n be the projective operator from $L^2(\mathbb{R},\,\varpi(x)dx)$ to $\mathcal{H}_n$. we construct an operator $T_{n}^{(k)}$ from $\mathcal{H}_n$ to itself as follows,

$$T_{n}^{(k)}=P_n\circ A_{x}^{k}, \tag{34}$$

where A_x is multiplication by x.

Now we consider the k-moment $M_{n}^{(k)}$ and $\hat{M}_{n}^{(k)}$ of probability density $\sigma_n(x)$ and $\hat{\sigma}_n(x)$ respectively.

Proposition 4.1.

$$M_{n}^{(k)}=\frac{Tr(T_{n}^{(k)})}{n\cdot(D_n)^{k/2}}. \tag{35}$$

Proof.

$$M_n^{(k)} = \int_{\mathbb{R}} x^k \sigma_n(x)\,dx = \frac{1}{n\cdot (D_n)^{k/2}} \sum_{j=0}^{n-1} \Big\langle x^k p_j,\, p_j \Big\rangle_{L^2(\varpi)}$$

$$= \frac{1}{n\cdot (D_n)^{k/2}} \sum_{j=0}^{n-1} \Big\langle A_x^k(p_j),\, P_n(p_j) \Big\rangle_{L^2(\varpi)}$$

$$= \frac{1}{n\cdot (D_n)^{k/2}} \sum_{j=0}^{n-1} \Big\langle T_n^{(k)}(p_j),\, p_j \Big\rangle_{L^2(\varpi)} = \frac{Tr(T_n^{(k)})}{n\cdot (D_n)^{k/2}}.$$

Lemma 4.2.

$$||A_x f||_{L^2(\varpi)} \le 3N_n ||f||, \quad \textit{for all } f \in \mathcal{H}_n, \tag{36}$$

where $N_n = \sup_{0\le i\le n-1} \{|\alpha_i|,\, |\beta_i|\}$.

Proof. Put $f = \sum_{i=0}^{n-1} c_i p_i$, then

$$||A_x f||^2 = ||\sum_{i=0}^{n-1} c_i x p_i||^2 = ||\sum_i c_i(\alpha_i p_{i+1} + \beta_i p_i + \gamma_i p_{i-1})||^2$$

$$= ||\sum_i (c_{i-1}\alpha_{i-1} + c_i\beta_i + c_{i+1}\gamma_{i+1})p_i||^2$$

$$= \sum_i |c_{i-1}\alpha_{i-1} + c_i\beta_i + c_{i+1}\alpha_i|^2$$

$$\le 3N_n^2 \sum_i (c_{i-1}^2 + c_i^2 + c_{i+1}^2) \le 9N_n^2 ||f||^2.$$

Therefore, $||A_x f|| \le 3N_n ||f||$.

Corollary 4.3.

$$||A_x^k f|| \le 3^k \Big(\prod_{j=0}^{k-1} N_{n+j}\Big) ||f||, \quad \textit{for all } f \in \mathcal{H}_n. \tag{37}$$

Proof. By the above proposition, it is obvious.

Proposition 4.4.

$$\hat{M}_n^{(k)} = M_n^{(k)} + I, \tag{38}$$

where

$$I \equiv \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=n-l}^{n-1} \Big\langle A_x^k(\hat{p}_j p),\, \hat{p}_j p \Big\rangle_{L^2(\varpi)} - \sum_{j=0}^{l-1} \Big\langle T_n^{(k)}(e_j^{(n)}),\, e_j^{(n)} \Big\rangle_{L^2(\varpi)} \Big).$$

Proof.

$$\hat{M}_n^{(k)} = \int_{\mathbb{R}} x^k \hat{\sigma}_n(x)\,dx = \frac{1}{n\cdot (D_n)^{k/2}} \int_{\mathbb{R}} x^k \hat{R}_n^1(x)\,dx$$

$$= \frac{1}{n\cdot (D_n)^{k/2}} \sum_{j=0}^{n-1} \int_{\mathbb{R}} x^k \hat{p}_j^2(x) p^2(x) \varpi(x)\,dx$$

$$= \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=0}^{n-l-1} \int_{\mathbb{R}} x^k \hat{p}_j^2(x) p^2(x)\varpi(x)\,dx + \sum_{j=0}^{l-1} \int_{\mathbb{R}} x^k (e_j^{(n)}(x))^2\,dx \Big)$$

$$+ \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=n-l}^{n-1} \int_{\mathbb{R}} x^k \hat{p}_j^2(x) p^2(x)\varpi(x)\,dx - \sum_{j=0}^{l-1} \int_{\mathbb{R}} x^k (e_j^{(n)}(x))^2\,dx \Big)$$

$$= \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=0}^{n-l-1} \Big\langle T_n^{(k)}(\hat{p}_j p),\, \hat{p}_j p \Big\rangle_{L^2(\varpi)} + \sum_{j=0}^{l-1} \Big\langle T_n^{(k)}(e_j^{(n)}),\, e_j^{(n)} \Big\rangle_{L^2(\varpi)} \Big)$$

$$+ \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=n-l}^{n-1} \Big\langle A_x^k(\hat{p}_j p),\, \hat{p}_j p \Big\rangle_{L^2(\varpi)} - \sum_{j=0}^{l-1} \Big\langle T_n^{(k)}(e_j^{(n)}),\, e_j^{(n)} \Big\rangle_{L^2(\varpi)} \Big)$$

$$= \frac{Tr(T_n^{(k)})}{n\cdot (D_n)^{k/2}} + I\ .$$

Then by proposition 4.1, we obtain the conclusion.

Theorem 4.5. *Under the exponential growth conditions,*

$$\lim_{n\to\infty} M_n^{(k)} = \lim_{n\to\infty} \hat{M}_n^{(k)}, \quad \textit{for any } k \in \mathbb{Z}^+. \tag{39}$$

Proof. By the exponential growth conditions and corollary 4.3, we see that there exist constants $C_1,\, C_2 > 0$ such that

$$\Big| \Big\langle A_x^k(\hat{p}_j p),\, \hat{p}_j p \Big\rangle_{L^2(\varpi)} \Big| \le ||A_x^k \hat{p}_j p|| \le 3^k N_{n+l+k-2}^k \le C_1 n^{kt} \tag{40}$$

and

$$\Big| \Big\langle T_x^{(k)}(e_j^{(n)}),\, e_j^{(n)} \Big\rangle_{L^2(\varpi)} \Big| \le ||P_n|| \cdot ||A_x^k e_j^{(n)}|| \le 3^k N_{n+k-1}^k \le C_2 n^{kt}. \tag{41}$$

Thus by proposition 4.4, as $n >> 1$,

$$|I| \le \frac{1}{n\cdot (D_n)^{k/2}} \Big(\sum_{j=n-l}^{n-1} C_1 n^{kt} + \sum_{j=0}^{l-1} C_2 n^{kt} \Big)$$

$$= \frac{(C_1 + C_2) l}{n} \cdot \frac{n^{kt}}{(D_n)^{k/2}} = O(\frac{1}{n}).$$

Whereupon we obtain

$$\lim_{n\to\infty} M_n^{(k)} = \lim_{n\to\infty} \hat{M}_n^{(k)}.$$

Remark. In the case of finite interval $(-s,\, s) \subset \mathbb{R}$, note that the operator A_x is bounded, so the inequalities (40) and (41) in the above proof are naturally valid.

Finally we have the following theorem which tells us that after the weight function $\varpi(x)$ is appended a polynomial multiplicative factor, the limit behavior of the normalized 1-level correlation function is unaffected in the weak sense.

Theorem 4.6. *Under the exponential growth conditions,*

$$\hat{\sigma}_n(x) \xrightarrow{w} \sigma(x), \quad n \to \infty,$$

where w means weak convergence.

Proof. It is completely analogous to the proof of theorem 3.3, here we omit it.

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1. School of mathematical science, Peking University, Beijing, 100871, P. R. China
E-mail address: `zdwang@pku.edu.cn`

2. School of mathematics and physics, Zhejiang Normal University, Jinhua city, Zhejiang Province, 321004, P. R. China
E-mail address: `yankh@zjnu.cn`